Combining Scattering Matrices

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ABSTRACT

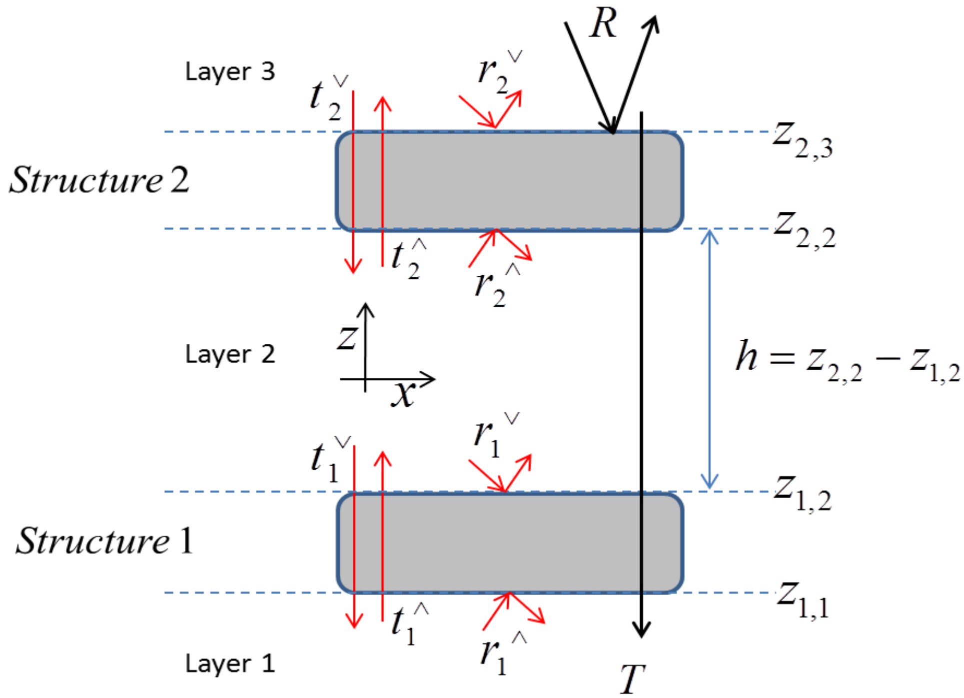
Formulae are derived for computing the total scattering matrix of a combination of individual scatterers (gratings, particles, surfaces, etc.) given the scattering matrices of the individual non-overlapping scatterers and their spatial relation to one another.

1 Background

Consider the structure shown in Figure 1 which consists of two individual scatterers labeled "Structure 1" and "Structure 2". The r and t matrices are the reflection and transmission scattering matrices, respectively, of the individual structures. We will define what we mean by a scattering matrix in detail below. We refer to them as matrices since they depend on both the incoming wave vector component in the x direction, β' and on the outgoing component, β, i.e. $r = r\left(\beta, \beta'\right)$ and $t = t\left(\beta, \beta'\right)$. Generally β and β' are continuous variables but for periodic structures they often can be treated as discrete, hence the term "matrices". An $r^{\vee}$ relates the value of the incoming field (propagating in the $-z$ direction) evaluated at the top surface of a structure to the outgoing field (propagating in the $+z$ direction) also evaluated at the top surface The $t^{\vee}$ matrix relates the value of the incoming field evaluated at the top surface to the outgoing field (propagating in the $-z$ direction) evaluated at the bottom surface. Similarly for $r^{\wedge}$ and $t^{\wedge}$ but with "top" and "bottom" exchanged. In sum, the superscript characters $\vee$ and $\wedge$ indicate whether the reflection and transmission is from the upper side or the lower side of a given structure, respectively. The R and T matrices are the net reflection and transmission matrices, respectively, of the combined structure. Here we just consider net reflection in region 3 and net transmission from region 3 to region 1.

Our purpose is to derive formulae for R and T in terms of the r and t matrices and the spatial relation between individual scatterers.

Here we explicitly consider the scattering of electromagnetic waves in two dimensions (2D), i.e., $\vec{\rho} = x\hat{x} + z\hat{z}$ where $\hat{x}$ and $\hat{z}$ are unit vectors in the x and y directions, respectively. But the results generalize directly to 3D and to any other type of scattering governed by a wave equation, e.g., sound waves, solutions to the time-independent Schrodinger equation, etc.. For TE polarization where the electric field is given by $\vec{E} = \phi\left(\vec{\rho}\right) e^{-i\omega t}\hat{y} = \phi\left(x, z\right) e^{-i\omega t}\hat{y}$ with the y axis into the page in Figure 1, $\hat{y}$ the unit vector in the y direction and t the time, Maxwells



equations reduce to

$$\left(\vec{\partial}^2 + n(\vec{\rho})^2 k^2\right)\phi(\vec{\rho}) = 0 \tag{1}$$

where $k = \omega/c$ with c the vacuum speed of light and $n(\vec{\rho})$ the spatially dependent index of refraction. The boundary conditions for TE polarization are: ϕ and $\hat{s}\cdot\vec{\partial}\phi$ are continuous across discrete jumps in the index n with $\hat{s}$ is the unit normal to the interface between the different index values and $\vec{\partial} = \hat{x}\partial_x + \hat{z}\partial_z$. For TM polarization where the magnetic field is given by $\vec{B} = \phi(\vec{r})\, e^{-i\omega t}\hat{y} = \phi(x,z)\, e^{-i\omega t}\hat{y}$ and we assume the magnetic permeability is constant over all space, Maxwells equations reduce to

$$\vec{\partial}\cdot\left(\frac{1}{n(\vec{\rho})^2}\vec{\partial}\phi(\vec{\rho})\right) + k^2\phi(\vec{\rho}) = 0 \tag{2}$$

with the boundary condtions: ϕ and $\hat{s}\cdot\vec{\partial}\left(\phi/n^2\right)$ are continuous across discrete jumps in the index. It should be noted that the boundary conditions follow directly from Eqs. 1 and 2 in the limit where $n(\vec{\rho})$ becomes step wise constant.

2 Reflection and Transmission Matrices of Isolated Structures

In the regions of the structures 1 and 2, i.e., between $z_{1,1}$ and $z_{1,2}$ and between $z_{2,2}$ and $z_{2,3}$ the index n is assumed to vary with both x and z. (Note for $z_{s,\ell}$ the first number indicates the structure, $s = 1, 2$, and the second the region number, $\ell = 1, 2.3$.) In the homogeneous layers above $z_{2,3}$, between $z_{1,2}$ and $z_{2,2}$ and below $z_{1,1}$ we take n to be spatially constant and so in those regions the solutions $\phi(\vec{r})$ to either Eq. 1 or Eq. 2 can be written as superpositions of plane waves in the form

$$\begin{aligned}\phi_\ell(\vec{\rho}) &= \phi_\ell^+(\vec{\rho}) + \phi_\ell^-(\vec{\rho}) \\ &\text{with} \\ \phi_\ell^p(\vec{\rho}) &= \int_{-\infty}^{+\infty} d\beta\ \exp\left[i\beta x + ip\gamma_i(\beta)(z - z_0)\right]\ \tilde{\phi}_\ell^p(\beta, z_0)\end{aligned} \tag{3}$$

Here $\gamma_i(\beta) = \sqrt{n_i^2 k^2 - \beta^2}$ where k_i is the value of k in region a. p takes the "values" $+$ and $-$ when a superscript an $+1$ and -1, respectively when multiplying γ_i. Note that for n_i real if $|\beta| < n_i k$ then the $\exp\left[i\gamma_i(\beta) z\right]$ is an oscillating function of z, i.e., propagating waves, whereas if $|\beta| > n_i k$ then $\gamma_i(\beta) = i\sqrt{\beta^2 - n_i^2 k^2}$ the $+$ waves ($-$ waves) decay in the $+z$ $(-z)$ directions, respectively, i.e., they are evanescent waves. For complex valued n_i this sharp disctinction is lost and the waves are a combination of both propagating and evanescent for all β. In the homogeneous regions propagating the field in the x and z directions amounts to multiplying $\tilde{\phi}_\ell^p$ by the phase factor $\exp\left[i\beta x + ip\gamma_i(\beta) z\right]$. The z_0 dependence is included in $\tilde{\phi}_\ell^p$ to indicate in which constant z plane is $\tilde{\phi}_\ell^p$ the Fourier transform in x of ϕ_ℓ^p with the z dependent phase factor equal to unity, viz.,

$$\tilde{\phi}_\ell^p(\beta, z_0) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \exp\left[-i\beta x\right] \phi_\ell^p(x, z = z_0) \tag{4}$$

To simplify notation define

$$\tilde{\phi}_{s,\ell}^p(\beta) = \tilde{\phi}_\ell^p(\beta, z_{s,\ell}) \tag{5}$$

and drop the limits of integration, which are always $-\infty$ to $+\infty$ on all β and x integrations. Given the above definition of the reflection and transmission matrices of the individual structures, i.e., when only one structure is present, they relate the $\tilde{\phi}_{s,\ell}^p$ to each other as follows

$$\begin{aligned}\tilde{\phi}_{1,1}^-(\beta) &= \int d\beta' t_1^\vee(\beta, \beta')\, \tilde{\phi}_{1,2}^-(\beta') + \int d\beta' r_1^\wedge(\beta, \beta')\, \tilde{\phi}_{1,1}^+(\beta') \\ \tilde{\phi}_{1,2}^+(\beta) &= \int d\beta' t_1^\wedge(\beta, \beta')\, \tilde{\phi}_{1,1}^+(\beta') + \int d\beta' r_1^\vee(\beta, \beta')\, \tilde{\phi}_{1,2}^-(\beta')\end{aligned} \tag{6}$$

for the Structure 1 and

$$\begin{aligned}\tilde{\phi}_{2,2}^-(\beta, z_2^\wedge) &= \int d\beta' t_2^\vee(\beta, \beta')\, \tilde{\phi}_{2,3}^-(\beta') + \int d\beta' r_2^\wedge(\beta, \beta')\, \tilde{\phi}_{2,2}^+(\beta') \\ \tilde{\phi}_{2,3}^+(\beta, z_2^\vee) &= \int d\beta' t_2^\wedge(\beta, \beta')\, \tilde{\phi}_{2,2}^+(\beta') + \int d\beta' r_2^\vee(\beta, \beta')\, \tilde{\phi}_{2,3}^-(\beta')\end{aligned}$$

for Structure 2. The r and t matrices are functions of an incoming β' and an outgoing β and the integral over β' is equivalent to matrix multiplication on the $\tilde{\phi}_{s,\ell}^p$ and $\tilde{\phi}_{s,\ell}^p$. Using the notation $\int d\beta' f(\beta, \beta')\, g(\beta') \equiv f \cdot g$ the above equations can be combined and abbreviated as

$$\begin{aligned}\tilde{\phi}_{s,\ell}^- &= t_s^\vee \cdot \tilde{\phi}_{s,\ell+1}^- + r_s^\wedge \cdot \tilde{\phi}_{s.\ell}^+ \\ \tilde{\phi}_{s,\ell+1}^+ &= t_s^\wedge \cdot \tilde{\phi}_{s,\ell}^+ + r_s^\vee \cdot \tilde{\phi}_{s,\ell+1}^-\end{aligned} \tag{7}$$

Note that $\tilde{\phi}_{s,\ell+1}^-$ and $\tilde{\phi}_{s,\ell}^+$ are incoming fields and hence are treated as given. To determine the individual r and t matrices requires solving Eqs. 1 and 2 in the inhomogeneous regions.

3 Combined Structure R and T matrices

To solve for R and T for the combined structure consider the net contribution to $\tilde{\phi}_{2,2}^-$. It is the sum of what is transmitted down through Structure 1, i.e., $t_2^\vee \cdot \tilde{\phi}_{2,3}^-$ and what is reflected off the bottom of Structure 1. But tracing where this reflected component comes from we find it is given by

$$
\begin{aligned}
& r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h) \cdot \tilde{\phi}_{2,2}^{-} \\
= & \int d\beta_4 d\beta_3 d\beta_2 d\beta_1 r_2^{\wedge}(\beta, \beta_4) \exp\left[i\gamma_2(\beta_4) h\right] \delta(\beta_4 - \beta_3) r_1^{\vee}(\beta_3, \beta_2) \exp\left[i\gamma_2(\beta_2) h\right] \delta(\beta_2 - \beta_1) \tilde{\phi}_{2,2}^{-}(\beta_1)
\end{aligned} \tag{8}
$$

Defining the matrix form of $\exp\left[i\gamma_2(\beta) h\right]$ as $p_2(h) = \exp\left[i\gamma_2(\beta) h\right] \delta\left(\beta - \beta'\right)$ where $\delta(\cdots)$ is a Dirac delta function and combining this with the transmitted portion gives

$$
\tilde{\phi}_{2,2}^{-} = t_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-} + r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h) \cdot \tilde{\phi}_{2,2}^{-} \tag{9}
$$

Solving for $\tilde{\phi}_{2,2}^{-}$ gives

$$
\tilde{\phi}_{2,2}^{-} = \left(I - r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h)\right)^{-1} \cdot t_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-} \tag{10}
$$

where I is the identity matrix, i.e., $\delta\left(\beta - \beta'\right)$.

Using the same approach consider the net contribution to $\tilde{\phi}_{2,3}^{+}$. It is given by what is reflected from the top of Structure 2, i.e., $r_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-}$ plus what is transmitted up through it from $z_{2,2}$. But tracing this back to $\tilde{\phi}_{2,2}^{-}$ and adding it to $r_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-}$ we get

$$
\begin{aligned}
\tilde{\phi}_{2,3}^{+} &= r_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-} + t_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h) \cdot \tilde{\phi}_{2,2}^{-} \\
&= \left(r_2^{\vee} + t_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h) \cdot \left(I - r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h)\right)^{-1} \cdot t_2^{\vee}\right) \cdot \tilde{\phi}_{2,3}^{-}
\end{aligned} \tag{11}
$$

Thus we obtain

$$
R = r_2^{\vee} + t_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h) \cdot \left(I - r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h)\right)^{-1} \cdot t_2^{\vee} \tag{12}
$$

Again using the same approach we find

$$
\begin{aligned}
\tilde{\phi}_{1,1}^{-} &= t_1^{\vee} \cdot p_2(h) \cdot \tilde{\phi}_{2,2}^{-} \\
&= t_1^{\vee} \cdot p_2(h) \cdot \left(I - r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h)\right)^{-1} \cdot t_2^{\vee} \cdot \tilde{\phi}_{2,3}^{-}
\end{aligned} \tag{13}
$$

and so

$$
T = t_1^{\vee} \cdot p_2(h) \cdot \left(I - r_2^{\wedge} \cdot p_2(h) \cdot r_1^{\vee} \cdot p_2(h)\right)^{-1} \cdot t_2^{\vee} \tag{14}
$$

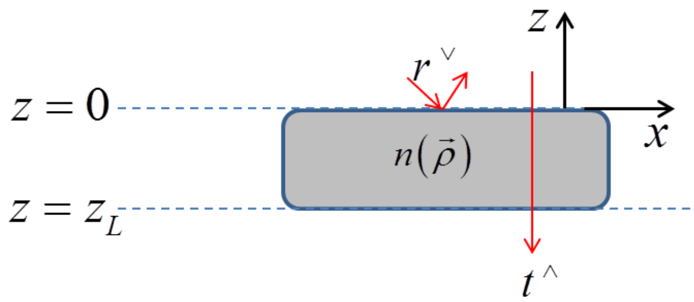
The dependence of p_2 on h allows for direct calculation of R and T as functions of the vertical spacing between the upper and lower structures. We now want to be able to shift the horizontal positions as well. For this we need to determine how the r and t matrices change when translated in x.

4 Translation in x

We can determine how translation in x affects $r^{\vee}$ and $t^{\vee}$ for an individual scatterer by relating them to the Greens function G for Eqs. 1 and 2. This derivation is the long way around but it also provides the relation between r and t matrices and G. We will only explicitly write out the $r^{\vee}$ case. For TE polarization the Greens function $G(\vec{\rho}, \vec{\rho}') = G(x, z, x', z')$ is defined by

$$
\left(\vec{\partial}^2 + k^2 n(\vec{\rho})^2\right) G(\vec{\rho}, \vec{\rho}') = \delta(\vec{\rho} - \vec{\rho}') \tag{15}
$$

with outgoing boundary conditions at spatial infinity. Take $n(\vec{\rho})$ to be constant for $z > 0$ and the same constant for $z < z_L < 0$ and to have arbitrary finite variation in the region $z_L < z < 0$, i.e., there is an arbitrary scattering



structure in the region $z_L < z < 0$. See Figure 2. Since the system is not translation invariant G depends independently on x, x' and z, z' and not on $x - x'$ and $z - z'$. Thus Fourier tranforming with respect to both $\vec{\rho}$ and $\vec{\rho}'$, G can be written as

$$G(\vec{\rho}, \vec{\rho}') = \frac{1}{(2\pi)^4} \int d\beta d\beta' d\gamma d\gamma' g(\beta, \gamma, \beta', \gamma') \exp\left[i(\beta x - \beta' x') + i(\gamma z - \gamma' z')\right] \tag{16}$$

where g satisfies

$$\left(-\beta^2 - \gamma^2 + k^2 n(-i\partial_\beta, -i\partial_\gamma)^2\right) g(\beta, \gamma, \beta', \gamma') = \delta(\beta - \beta')\,\delta(\gamma - \gamma') \tag{17}$$

Note that it follows from this that $\left(\vec{\partial}'^2 + k^2 n(\vec{\rho}')^2\right) G(\vec{\rho}, \vec{\rho}') = \delta(\vec{\rho} - \vec{\rho}')$ since we can switch the dummy integration variables in Eq 16, i.e., G is symmetric in $\vec{\rho}$ and $\vec{\rho}'$.

Now shift the scattering object in the x direction by a distance s, i.e., $n(x, z) \to n(x - s, z)$. We want to find a Greens function G' that satisfies

$$\left(\partial_x^2 + \partial_z^2 + k^2 n(x - s, z)^2\right) G'(x, z, x', z') = \delta(\vec{\rho} - \vec{\rho}') \tag{18}$$

But if we shift both x and x' in G by s then it will satisfy

$$\begin{aligned} \delta(\vec{\rho} - \vec{\rho}') &= \left(\partial_{x-s}^2 + \partial_z^2 + k^2 n(x - s, z)^2\right) G(x - s, z, x' - s, z') \\ &= \left(\partial_x^2 + \partial_z^2 + k^2 n(x - s, z)^2\right) G(x - s, z, x' - s, z') \end{aligned} \tag{19}$$

and so $G'(x, z, x', z') = G(x - s, z, x' - s, z')$.

To determine r and how it changes with the shift s write the solution to

$$\left(\vec{\partial}^2 + k^2 n(\vec{\rho})^2\right) \phi(\vec{\rho}) = 0 \tag{20}$$

with an incoming field $\phi^-(\vec{\rho})$ from $z > 0$ as

$$\begin{aligned} \phi(\vec{\rho}) &= \phi^-(\vec{\rho}) + \phi^+(\vec{\rho}) \\ &= \int d\beta' e^{i\beta x - i\gamma(\beta) z} + \int d\beta' r^{\vee}(\beta', \beta)\, e^{i\beta x + i\gamma(\beta) z} \end{aligned} \tag{21}$$

We take incoming field $\phi^-(\vec{\rho})$ to satisfy

$$\left(\vec{\partial}^2 + n^2 k^2\right) \phi^-(\vec{\rho}) = 0 \tag{22}$$

with $n = n(x, z > 0) =$ constant, it can be written as

$$\phi^-(\vec{\rho}) = \int d\beta \exp\left[i\beta x - i\gamma(\beta) z\right] \tilde{\phi}^-(\beta) \tag{23}$$

where $\gamma(\beta) = \sqrt{n^2k^2 - \beta^2}$. In the region $z > 0$ the reflected/scattered field $\phi^+(x, z > 0)$ can be written as

$$\phi^+(x, z) = \int d\beta d\beta' \exp\left[i\beta x + i\gamma(\beta) z\right] r^\vee\left(\beta, \beta'\right) \tilde{\phi}^-\left(\beta'\right) \tag{24}$$

The fact that we can write $\phi^+(\vec{\rho})$ in this form follows from the linearity of the wave equation, Eq. 20. Substitute the first line of Eq 21 into Eq 20 and use Eq. 15 to get

$$\left(\vec{\partial}^2 + k^2 n(\vec{\rho})^2\right)\left(\phi^-(\vec{\rho}) + \phi^+(\vec{\rho})\right) = 0 \tag{25}$$

After using Eq. 22, Eq. 15, and the symmetry of G we get

$$\phi^+(\vec{\rho}) = \int d^2\rho' G(\vec{\rho}, \vec{\rho}') k^2 \left(n^2 - n(\vec{\rho}')^2\right) \phi^-(\vec{\rho}') \tag{26}$$

Setting the right hand sides of Eq. 24 and Eq. 26 equal, evaluating both at $z = 0_+$ and letting $k^2\left(n^2 - n(\vec{\rho}')^2\right) \equiv V(\vec{\rho}') = 0$ we have

$$\int d\beta d\beta' \exp\left[i\beta x\right] r^\vee\left(\beta, \beta'\right) \tilde{\phi}^-\left(\beta'\right) = \int dx' \int_{z_L}^{0} dz' G(x, 0_+, \vec{\rho}') V(\vec{\rho}') \phi^-(\vec{\rho}') \tag{27}$$

Fourier transforming both sides of Eq. 27 and writing $\phi^-(\vec{\rho}')$ in terms of its Fourier transform gives

$$\begin{aligned}\int d\beta' r^\vee\left(\beta, \beta'\right) \tilde{\phi}^-\left(\beta'\right) &= \int \frac{dxdx'}{2\pi} \int_{z_L}^{0} dz' e^{-i\beta x} G(x, 0_+, x', z') V(x', z') \int d\beta' e^{i\beta' x' - i\gamma(\beta')z'} \tilde{\phi}^-\left(\beta'\right) \\ &= \int d\beta' \int \frac{dxdx'}{2\pi} \int_{z_L}^{0} dz' e^{-i\beta x} G(x, 0_+, x', z') V(x', z') e^{i\beta' x' - i\gamma(\beta')z'} \tilde{\phi}^-\left(\beta'\right)\end{aligned} \tag{28}$$

Since this must hold for arbitrary $\tilde{\phi}^-(\beta)$ we have

$$r^\vee\left(\beta, \beta'\right) = \int \frac{dxdx'}{2\pi} \int_{z_L}^{0} dz' e^{-i\beta x} G(x, 0_+, x', z') V(x', z') e^{i\beta' x' - i\gamma(\beta')z'} \tag{29}$$

Shifting the position of the scattering object by s in the x direction and use the result given above, Eq 19, for the corresponding Greens function gives

$$\begin{aligned}r^{\vee\prime}\left(\beta, \beta'\right) &= \int \frac{dxdx'}{2\pi} \int_{z_L}^{0} dz' e^{-i\beta x} G(x - s, 0_+, x' - s, z') V(x' - s, z') e^{i\beta' x' - i\gamma(\beta')z'} \\ &= \int \frac{dxdx'}{2\pi} \int_{z_L}^{0} dz' e^{-i\beta(x+s)} G(x, 0_+, x', z') V(x', z') e^{i\beta'(x'+s) - i\gamma(\beta')z'} \\ &= e^{-i\beta s} r^\vee\left(\beta, \beta'\right) e^{i\beta' s}\end{aligned} \tag{30}$$

which makes perfect sense.

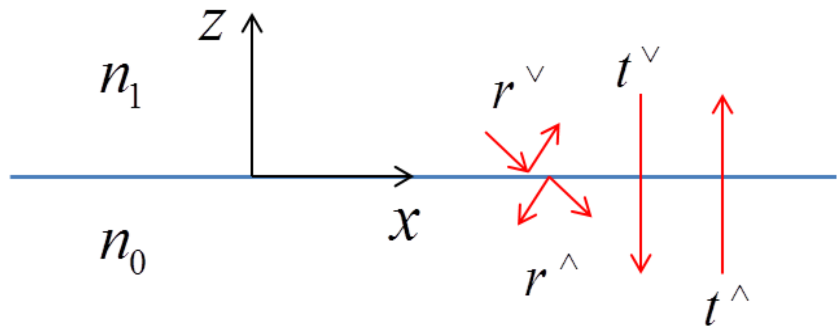
For TM polarization the Greens function is defined by

$$\left(\vec{\partial} \cdot \frac{1}{n(\vec{\rho})^2} \vec{\partial} + k^2\right) G(\vec{\rho}, \vec{\rho}') = \delta(\vec{\rho} - \vec{\rho}') \tag{31}$$

If we shift the structure in the x direction by s then as for TE polarization we find $G'(\vec{\rho}, \vec{\rho}') = G(x - s, z, x' - s, z')$ satisfies the shifted version of Eq 31 as for the TE case and so the same relation, Eq. 30 holds for TE polarization, which is what one would expect.

For transmission the same result holds for both polarizations

$$t^{\vee\prime}\left(\beta, \beta'\right) = e^{-i\beta s} t^\vee\left(\beta, \beta'\right) e^{i\beta s} \tag{32}$$



5 Test Case: r in terms of G

Test Eq 29 on a case which can be evaluated analytically. Consider the case where the index of refraction is homogeneous for both $z > 0$ and $z < 0$ but with value n_1 for $z > 0$ and n_0 for $z < 0$ as shown in Figure 3.

For TE polarization (electric field in the y direction) $E_y \equiv \phi$ satisfies the following boundary conditions at $z = 0$

$$\begin{aligned} \phi_1(x,0) &= \phi_0(x,0) \\ (\partial_z \phi_1)_{z=0} &= (\partial_z \phi_0)_{z=0} \end{aligned} \tag{33}$$

where ϕ_1 (ϕ_0) is the field in the region $z > 0$ $(z < 0)$. Since the structure is homogeneous the fields can built out of plane waves

$$\exp\left[i\beta x \pm i\gamma_\ell(\beta) z\right] \tag{34}$$

where $\gamma_\ell(\beta) = \sqrt{n_\ell{}^2 k^2 - \beta^2}$ with $\ell = 0, 1$ indicating the $z < 0$ and $z > 0$ half-spaces, respectively. Using the boundary conditions we find that

$$\begin{aligned} r^{\vee}(\beta) &= -r^{\wedge}(\beta) = \frac{\gamma_1(\beta) - \gamma_0(\beta)}{\gamma_1(\beta) + \gamma_0(\beta)} \\ t^{\vee}(\beta) &= 1 + r^{\vee}(\beta) = \frac{2\gamma_1(\beta)}{\gamma_1(\beta) + \gamma_0(\beta)} \\ t^{\wedge}(\beta) &= 1 + r^{\wedge}(\beta) = \frac{2\gamma_0(\beta)}{\gamma_1(\beta) + \gamma_0(\beta)} \end{aligned} \tag{35}$$

Translation invariance in the x direction means the Greens function can be written in the form

$$G(x - x', z, z') = \int \frac{d\beta}{2\pi} e^{i(\beta - \beta')x} g(\beta, z, z') \tag{36}$$

where g satifies

$$\left(\partial_z{}^2 + n(z)^2 k^2 - \beta^2\right) g(\beta, z, z') = \delta(z - z') \tag{37}$$

where

$$n(z) = \begin{cases} n_1 \text{ for } z > 0 \\ n_0 \text{ for } z < 0 \end{cases} \tag{38}$$

Fundamental solutions to

$$\left(\partial_z{}^2 + n(z)^2 k^2 - \beta^2\right) \phi = 0 \tag{39}$$

which satisfy the boundary conditions with incoming waves from $z=+\infty$ and $z=-\infty$ can be written as

$$\begin{aligned}\phi^{+}(\beta,z) &= \begin{cases}\exp\left[i\beta x-i\gamma_1 z\right]+r^{\vee}\exp\left[i\beta x+i\gamma_1 z\right] & \text{for } z>0\\ t^{\vee}\exp\left[i\beta x-i\gamma_0 z\right] & \text{for } z<0\end{cases}\\ \phi^{-}(\beta,z) &= \begin{cases}t^{\wedge}\exp\left[i\beta x+i\gamma_1 z\right] & \text{for } z>0\\ \exp\left[i\beta x+i\gamma_0 z\right]+r^{\wedge}\exp\left[i\beta x-i\gamma_0 z\right] & \text{for } z<0\end{cases}\end{aligned} \tag{40}$$

where the + and − superscripts on ϕ correspond to incoming waves from $z=+\infty$ and $z=-\infty$, respectively. Arbitrary field configurations can be built out of linear combinations of the $\phi^{\pm}$. In particular, the Greens function can be written as

$$G\left(x-x',z,z'\right)=\int\frac{d\beta}{2\pi}e^{i\beta\left(x-x'\right)}\frac{\theta\left(z-z'\right)\phi^{-}(\beta,z)\,\phi^{+}\left(\beta,z'\right)+\theta\left(z'-z\right)\phi^{+}(\beta,z)\,\phi^{-}\left(\beta,z'\right)}{W} \tag{41}$$

where $\theta(X)=1$ for $X>0$ and it 0 otherwise, i.e., $\delta(x)=\partial_x\theta(x)$. W is the Wronskian

$$\begin{aligned}W(\beta) &= \left(\partial_z^{-}\phi(\beta,z)\right)\phi^{+}(\beta,z)-\left(\partial_z^{+}\phi(\beta,z)\right)\phi^{-}(\beta,z)\\ &= 2i\gamma_1(\beta)\,t^{\wedge}(\beta)\end{aligned} \tag{42}$$

which for solutions to Eq. 39 is constant in z.

Substituting into Eq. 29 we get

$$\begin{aligned}r^{\vee}\left(\beta,\beta'\right) &= \int\frac{dxdx'}{2\pi}\int_{-\infty}^{0}dz'e^{-i\beta x+i\beta'x'-i\gamma_1\left(\beta'\right)z'}k^2\left(n_1{}^2-n_0{}^2\right)G\left(x-x',0_{+},z'\right)\\ &= k^2\left(n_1{}^2-n_0{}^2\right)\int\frac{d\beta''}{2\pi}\int\frac{dxdx'}{2\pi}\int_{-\infty}^{0}dz'e^{-i\beta x+i\beta'x'-i\gamma_1\left(\beta'\right)z'}e^{i\beta''\left(x-x'\right)}g\left(\beta'',0_{+},z'\right)\\ &= k^2\left(n_1{}^2-n_0{}^2\right)\int d\beta''\delta\left(\beta'-\beta''\right)\delta\left(\beta-\beta''\right)\int_{-\infty}^{0}dz'e'^{-i\gamma_1\left(\beta'\right)z'}\frac{t^{\wedge}\left(\beta''\right)t^{\vee}\left(\beta''\right)e^{-i\gamma_0\left(\beta''\right)z'}}{2i\gamma_1\left(\beta''\right)t^{\wedge}\left(\beta''\right)}\\ &= k^2\left(n_1{}^2-n_0{}^2\right)\delta\left(\beta-\beta'\right)\frac{t^{\vee}(\beta)}{2i\gamma_1(\beta)}\int_{-\infty}^{0}dz'e'^{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)z'}\\ &= k^2\left(n_1{}^2-n_0{}^2\right)\delta\left(\beta-\beta'\right)\frac{1}{i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)}\frac{1}{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)}\\ &= \frac{\gamma_1(\beta)-\gamma_0(\beta)}{\gamma_1(\beta)+\gamma_0(\beta)}\delta\left(\beta-\beta'\right)\\ &= r^{\vee}(\beta)\,\delta\left(\beta-\beta'\right)\end{aligned}$$

which is correct, $r^{\vee}(\beta)\,\delta\left(\beta-\beta'\right)$ is the matrix form of $r^{\vee}(\beta)$. The matrix is diagonal since for reflection off a planar surface $\beta=\beta'$.

In the above derivation we have used the fact that $k^2\left(n_1{}^2-n_0{}^2\right)=\gamma_1(\beta)^2-\gamma_0(\beta)^2=\left(\gamma_1(\beta)-\gamma_0(\beta)\right)\left(\gamma_1(\beta)+\gamma_0(\beta)\right)$ and

$$\begin{aligned}\int_{-\infty}^{0}dz'e'^{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)z'} &= \lim_{\varepsilon\to 0}\int_{-\infty}^{0}dz'e'^{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)z'+\varepsilon z'}\\ &= PV\left(\frac{1}{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)}\right)-i\pi\delta\left(\gamma_1(\beta)+\gamma_0(\beta)\right)\end{aligned}$$

But note that since $\gamma_1(\beta)+\gamma_0(\beta)$ is never zero we can ignore the delta-function and the principal value (PV) and just write

$$\int_{-\infty}^{0}dz'e'^{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)z'}=\frac{1}{-i\left(\gamma_1(\beta)+\gamma_0(\beta)\right)}$$

as done above.