Acoustic Pulses

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Acoustic Wave Equation (compressible gas)

$$\left(\vec{\partial}^2 - \frac{1}{c^2}\partial_t^2\right)\phi\left(\vec{r},t\right) = 0 \tag{1}$$

Here ϕ is the velocity potential, i.e., $\vec{\partial}\phi\left(\vec{r},t\right) = \vec{v}\left(\vec{r},t\right)$ where $\vec{v}\left(\vec{r},t\right)$ is the velocity of the gas atoms/molecules at position $\vec{r}$ and time t. c is the speed of sound.

Consider the 1+1 dimesional problem where an infinite inflexible movable "wall" is placed parallel to the yz plane with it's position along the x axis given by $X\left(t\right)$. Let the gas fill the semi-infiinite half space to the right of the wall, i.e., $x > X\left(t\right)$. When the wall speed $\partial_t X\left(t\right)$ is much less than the speed of sound the atoms/molecules just to the right of the wall must move with the speed of the wall, i.e. $v\left(X\left(t\right),t\right) = \partial_t X\left(t\right)$. This is only one boundary condition, nominally we need two. But since the wave equation can be factorized as $\left(\partial_x - \partial_t/c\right)\left(\partial_x + \partial_t/c\right)\phi = 0$ and we can solve either $\left(\partial_x - \partial_t/c\right)\phi = 0$ or $\left(\partial_x + \partial_t/c\right)\phi = 0$ to get waves/pulses moving in the $-x$ or the $+x$ direction, respectively we only need one boundary condition which is $\left(\partial_x\phi\right)\left(X\left(t\right),t\right) = v\left(X\left(t\right),t\right) = \partial_t X\left(t\right)$.

For $\partial_t X\left(t\right) << c$ we have the following approximate solution

$$\phi\left(x,t\right) = -cX\left(t - \frac{x - X\left(t\right)}{c}\right) \tag{2}$$

First show that this satisfies the wave equation up to terms of order $\partial_t X/c$. Let $\xi\left(x,t\right) = t - \left(x - X\left(t\right)\right)/c$, then

$$\begin{aligned}\left(\partial_x + \frac{1}{c}\partial_t\right)\left(-cX\left(\xi\left(x,t\right)\right)\right) &= -c\left(\partial_\xi X\right)\left(\partial_x\xi\right) + \frac{1}{c}\left(-c\right)\left(\partial_\xi X\right)\left(\partial_t\xi\right) \\ &= -c\left(\partial_\xi X\right)\left(-\frac{1}{c}\right) - \left(\partial_\xi X\right)\left(1 + \frac{\partial_t X\left(t\right)}{c}\right)\end{aligned}$$

But given $\partial_t X\left(t\right)/c << 1$ we have

$$\left(\partial_x + \frac{1}{c}\partial_t\right)\left(-cX\left(\xi\left(x,t\right)\right)\right) = \left(\partial_x + \frac{1}{c}\partial_t\right)\left(-cX\left(t - \frac{x - X\left(t\right)}{c}\right)\right) \cong 0 \tag{3}$$

Show that it satsifies the boundary conditions.

$$\left.\partial_x\phi\left(x,t\right)\right|_{x=X(t)} = \left.\partial_\xi X\left(\xi\left(x,t\right)\right)\right|_{x=X(t)} = \left(\partial_\xi X\right)\left(t\right) = \partial_t X\left(t\right) \tag{4}$$

The density variation $\delta\rho\left(x,t\right)$ about a constant background density, ρ_0, is related to the speed of the molecules via

$$\frac{v\left(x,t\right)}{c} = \frac{\delta\rho\left(x,t\right)}{\rho_0} \tag{5}$$

and so

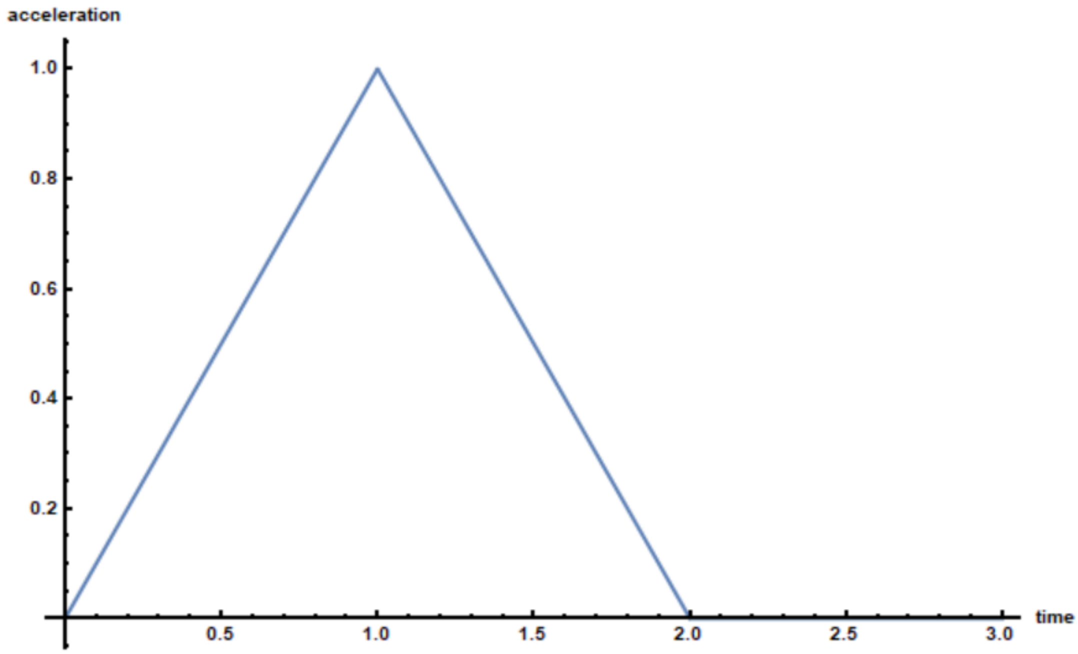
$$\frac{\delta\rho\left(x,t\right)}{\rho_0} = \frac{\partial_x\phi\left(x,t\right)}{c} = -\partial_x X\left(t - \frac{x - X\left(t\right)}{c}\right) \tag{6}$$

For the wall moving at a constant speed, $X\left(t\right) = Vt$, we have

$$\frac{\delta\rho\left(x,t\right)}{\rho_0} = -\partial_x\left(V \times \left(t - \frac{x - Vt}{c}\right)\right) = \frac{V}{c} \tag{7}$$

All molecules are moving at the same speed, $\partial_x v\left(x,t\right) = 0$. In order to generate acoustic waves or pulses the wall must change speed, i.e., accelerate.

Consider the case where the wall starts with zero speed and accelerates up to a finite speed. For a step function jerk profile, the acceleration, velocity and position look as follows. NOTE: jerk is the derivative of acceleration with respect to time and so a step function jerk induces a linear time dependence of the acceleration.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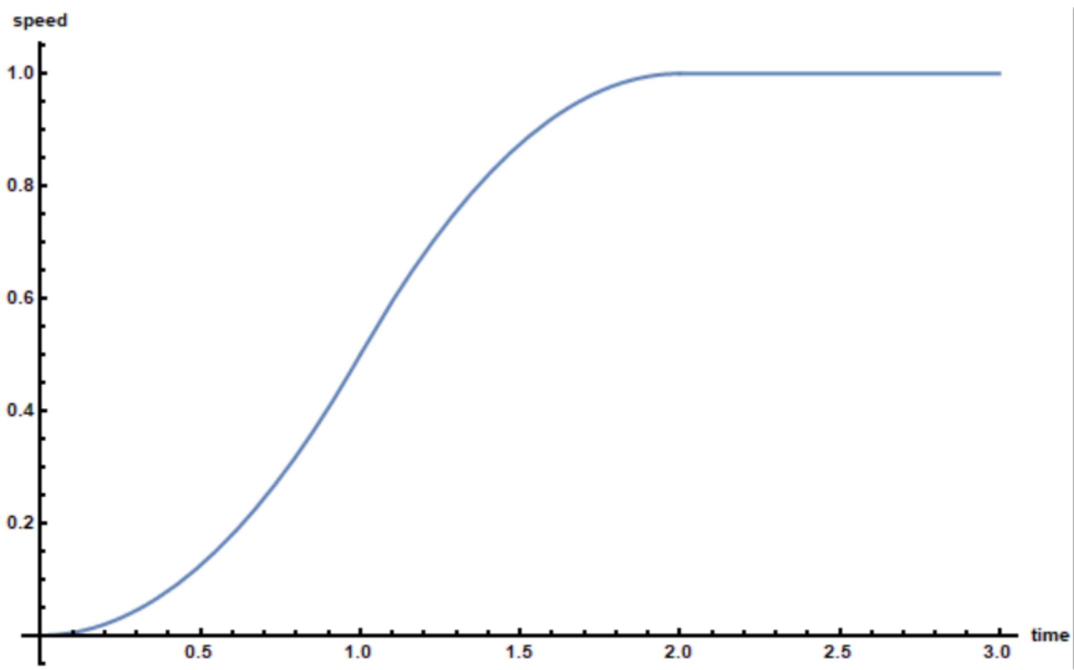
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
time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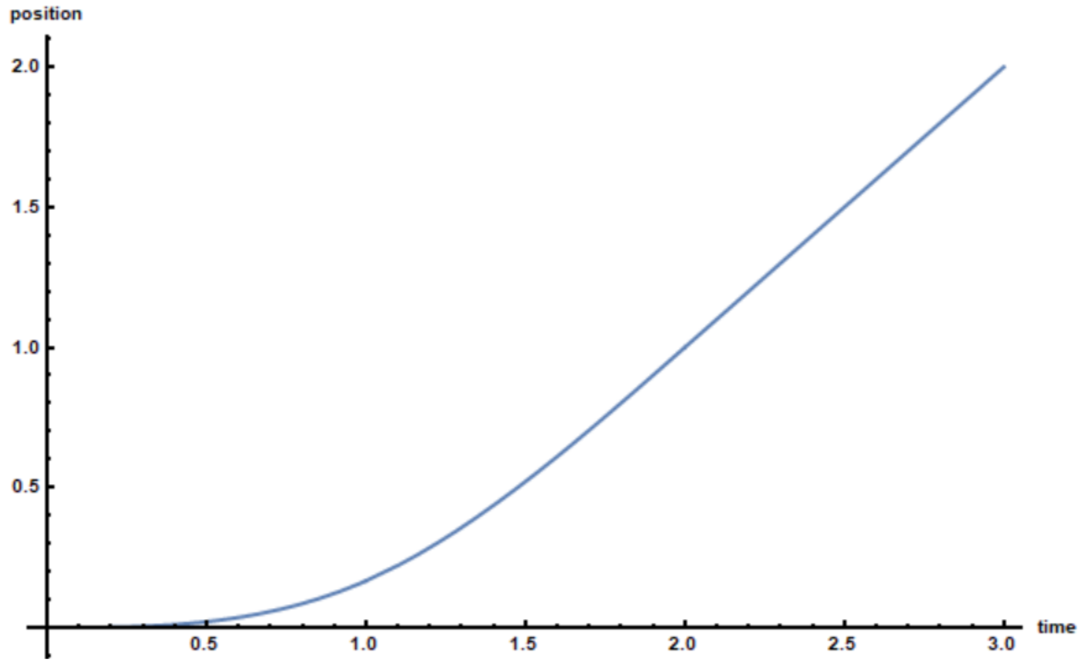
time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1.5
2.0
2.5
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time